Substituting from the condition $nk^{2(n-1)} = 1$ we find

$$\frac{d^2T}{dk^2} = -\frac{d(n-1)}{k}\left(\frac{n-1}{a\log n}\right)^{1/2}.$$

Since this result is negative ($n > 1$) a true maximum obtains.

Case (b). $1 > n > 0$.

Then $\overline{FP} < \overline{FQ}$. Proceeding as before we get

$$T' = -\int_{k^n d}^{kd}\left(2a\log\frac{d}{x}\right)^{-1/2}dx,$$

where T' symbolizes the interval of time consumed in going from Q to P. Consequently $nk^{2(n-1)} = 1$ and

$$\frac{d^2T'}{dk^2} = -\frac{d(1-n)}{k}\left(\frac{1-n}{-a\log n}\right)^{1/2}.$$

Since $n - 1$ is now negative, the second derivative is also negative, and hence the condition for a maximum is again fulfilled.

Remark: In making the final reductions in case (b) attention has to be paid to the fact that $\log n$ is negative as well as $n - 1$.

Also solved by A. H. Wilson, H. N. Carleton, H. Polish, J. A. Caparo, Elijah Swift, Paul Capron, and the Proposer.

NUMBER THEORY.

209. (March, 1914.) Proposed by R. D. CARMICHAEL, University of Illinois.

Prove that the difference of the sixth powers of two integers cannot be the square of an integer.

Solution by Elijah Swift, University of Vermont.

In Carmichael's Diophantine Analysis, pp. 70, 71, the impossibility of the equation $x^3 + y^3 = 2^n z^3$ is proved. (The proof is valid whether y is positive or negative.)

We are to prove the impossibility of the equation $a^6 - b^6 = c^2$. We know that if $x^2 + y^2 = z^2$, then $x = 2mn$, $y = m^2 - n^2$, $z = m^2 + n^2$. (Loc. cit., p. 10.) We assume, of course, that a, b, c are all prime to each other; so are x, y, z also. Two cases present themselves:

(I) $\quad a^3 = m^2 + n^2, \quad b^3 = m^2 - n^2, \quad c = 2mn.$

(II) $\quad a^3 = m^2 + n^2, \quad b^3 = 2mn, \quad c = m^2 - n^2.$

Case I. Of the two integers, m and n, one is odd and the other even. Also these numbers are prime to each other. Consequently $m + n$ and $m - n$ are prime to each other, and since their product is a cube, each of them must be a cube also. If we set them equal to α^3 and β^3 respectively

$$m + n = \alpha^3, \quad m - n = \beta^3, \quad a^3 = m^2 + n^2 = \frac{\alpha^6 + \beta^6}{2}, \quad \text{or} \quad \alpha^6 + \beta^6 = 2a^3,$$

the impossibility of which was stated in the first paragraph.

Case II. Assume m even; the proof will apply equally well to the other case, n even. Since $2mn$ is a cube, $2m$ and n must each of them be a cube. Setting them equal to $8\alpha^3$ and β^3, we have

$$m = 4\alpha^3, \quad n = \beta^3, \quad a^3 = 16\alpha^6 + \beta^6 \quad \text{or} \quad 2^4\alpha^6 = a^3 - \beta^6,$$

an equation which was proved to be impossible. Hence the equation $a^6 - b^6 = c^2$ cannot hold.

219. (June, 1914.) Proposed by R. D. CARMICHAEL, University of Illinois.

Determine whether it is possible for a polygon to have the number of its diagonals equal to a perfect fourth power.

Solution by Elijah Swift, University of Vermont.

If the number of sides of a polygon is n, the number of its diagonals is $n(n-3)/2$, which by the conditions of the problem must be a perfect fourth power.

The fourth power of any number is divisible by 16, if the number is even, by 16 with remainder 1 if the number is odd.

We consider two cases: (I) n not divisible by 3, (II) n divisible by 3.

Case I. If n is even, $n - 3$ is odd, and the two are prime to each other. Since their product is twice a fourth power, $n - 3$ is a fourth power. Consequently $n - 3$ is divisible by 16 with remainder 1, by n with remainder 4, and by $n/2$ with remainder 2 or 10; in neither case can $n/2$ be a fourth power, so that $n(n - 3)/2$ cannot be a fourth power either.

If n is odd, it must be a fourth power by the same reasoning as before. $n - 3$ has the remainder 14, and $(n - 3)/2$, has the remainder 7 or 15, when divided by 16; so that $(n - 3)/2$ cannot be a fourth power, nor can $n(n - 3)/2$.

Case II. If n is divisible by 3, $n - 3$ is also, and so is the fourth power, so that we have the equation $n(n - 3) = 2\cdot 3^4\cdot k^4$. Either n or $n - 3$ is divisible by 3^3 and the other one by 3. Calling the one $3^3\beta$ the other will be $3^3\beta \pm 3$, the sign depending on which is divisible by 3^3. Our equation takes the form $3^3\beta(3^3\beta \pm 1) = 2\cdot 3^4\cdot k^4$, or $\beta(3^2\beta \pm 1) = 2\cdot k^4$.

If β is odd, it must be a fourth power, and $(3^2\beta \pm 1)$ twice a fourth power. But if $\beta \equiv 1 \pmod{16}$, $3^2\beta \pm 1 \equiv 8$ or $10 \pmod{16}$, so that it could not be twice a fourth power.

If β is even, $\beta/2$ and $3^2\beta \pm 1$ must both be fourth powers. Here, again, two cases are possible: $\beta/2$ even and $\beta/2$ odd. In the latter case $\beta/2 \equiv 1 \pmod{16}$, or $\beta \equiv 2 \pmod{16}$, so that $3^2\beta \pm 1 \equiv 3$ or $1 \pmod{16}$, from which we deduce at once that only the $-$ sign is admissible. But $3^2\beta - 1 =$ a fourth power, and any fourth power if divided by 3 must yield the remainder $+1$ or 0. Consequently $3^2\beta - 1$ cannot be a fourth power, if $\beta/2$ is odd.

It remains to consider the case $\beta/2$ even. Just as before, we see that $\beta/2$ and $3^2\beta \pm 1$ must be fourth powers. Calling these $16b^4$ and a^4 respectively, (since $\beta/2$ is even, it must be the fourth power of an even integer) we derive the equations $288b^4 \pm 1 = a^4$. Since $a^4 \equiv 1 \pmod{16}$. the $+$ sign only is admissible. We now proceed to show the impossibility of this last equation,

Writing it in the form $288b^4 = a^4 - 1$, we see that for the right hand side to be divisible by 32, a must be of the form $8\lambda \pm 1$. Substituting this value for a and factoring the right-hand side, the equation becomes

$$288b^4 = (64\lambda^2 \pm 16\lambda + 2)(8\lambda \pm 2)8\lambda,$$

or, after division by 32,

$$9b^4 = (32\lambda^2 \pm 8\lambda + 1)(4\lambda \pm 1)\lambda.$$

It is easy to see that the three factors on the right are prime to each other: in fact

$$(32\lambda^2 \pm 8\lambda + 1) - (4\lambda \pm 1)^2 = 16\lambda^2,$$

so that any common factor of the first two must be a factor of $16\lambda^2$. Since the product of these three is a perfect square, each of them must be a square, and we have the three equations

$$\lambda = c^2,$$

$$4\lambda^2 \pm 1 = d^2,$$

$$32\lambda^2 \pm 8\lambda + 1 = e^2.$$

From the first two of these we deduce

$$4c^2 \pm 1 = d^2,$$

which is impossible, for the squares of two integers cannot differ by unity.

As we have examined all possible cases, we conclude that no polygon can have the number of its diagonals equal to a perfect fourth power.

222. (October, 1914.) Proposed by A. H. HOLMES, Brunswick, Me.

Find rational values for m and n such that $(m^2 + 1)/m^2 + (n^2 + 1)/n^2$ may be the square of an integer.

SOLUTION BY ELIJAH SWIFT, University of Vermont.

Let $m = b/a$, $n = d/c$, where a, b, c, d, are all integers, and where a is prime to b, and c to d. Substituting these values, $\dfrac{b^2 + a^2}{b^2} + \dfrac{c^2 + d^2}{d^2}$ must be the square of an integer. If it is to be an integer at all, clearly b must equal d, so that we have the equation

$$\frac{a^2 + c^2 + 2b^2}{b^2} = \text{a perfect square},$$